Josephson effect between Bose condensates

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I. INTRODUCTION

Since its theoretical prediction in 1962 [1], the Josephson effect has played a major role in the physics and technology of superconductors [2]. The physics of the Josephson effect becomes manifest when a weak link is created (*e.g.* by tunneling or through a point contact) between two systems which have undergone some type of gauge symmetry breaking. In fact, the Josephson effect has also been observed in superfluids [3], and, because of the profound analogies, its observation in the recently achieved [4–7] Bose condensed atomic gases is generally expected. A system whose gauge symmetry has been spontaneously broken can be described by an order parameter that behaves in many respects like a macroscopic wave function $\Psi(\mathbf{r})$. In the simplest cases, the order parameter reduces to a complex scalar, $\Psi = \sqrt{\rho}e^{i\phi}$, where $\rho(\mathbf{r})$ is the "superfluid density" and $\phi(\mathbf{r})$ is the phase. For Bose-Einstein condensates of dilute alkali gases, $\Psi(\mathbf{r})$ is the wave function of the macroscopically occupied one-atom state.

In his epoch making paper [1], Josephson predicted that, between two weakly connected superconductors of phases φ_1 and φ_2, a non-dissipative particle (Cooper pair) current flows between them whose value is

$$I(\varphi) = I_c \sin\varphi, \tag{1}$$

where I_c is the critical current and $\varphi = \varphi_2 - \varphi_1$ is the relative phase [$\phi(\mathbf{r})$ is assumed to be approximately uniform within each superconductor]. He also predicted that, in the presence of a nonzero chemical potential difference $\mu = \mu_2 - \mu_1$, the relative phase rotates as

$$\dot{\varphi} = -\mu/\hbar \tag{2}$$

The Josephson relations can be obtained from very general considerations and apply to any pair of weakly linked systems that can be described by macroscopic wave functions [8]. They can be obtained as the equations of motion of the "pendulum Hamiltonian"

$$H(\varphi, N) = E_J(1-\cos\varphi) + \frac{1}{2}E_C N^2, \tag{3}$$

where $E_J = \hbar I_c$ is the Josephson coupling energy, $N = (N_2 - N_1)/2$ is the number of transferred particles, and $E_C \equiv \partial\mu/\partial N$ is the capacitive energy due to interactions. In the absence of external constraints, $\mu = E_C N$. In a superconducting link, the critical electric current is $2eI_c$ and the capacitive (or "charging") energy is $E_C = (2e)^2/2C$, where C is the electrostatic capacitance. For trapped BEC's, E_C can be obtained to a good accuracy from the Thomas-Fermi (TF) calculation [9] of the chemical potential [see eq. (23)].

When both E_C and k_BT are $\ll E_J$, the Josephson pendulum Hamiltonian can be approximated as a harmonic oscillator,

$$H(\varphi, N) \simeq \frac{1}{2}E_J\varphi^2 + \frac{1}{2}E_C N^2, \tag{4}$$

whose frequency

$$\omega_{JP} = \sqrt{E_J E_C}/\hbar \tag{5}$$

is called the Josephson plasma frequency.

Given the intimate relationship between eqs. (1-2) and eq. (3), a derivation of the pendulum Hamiltonian (in particular of the $-\cos\varphi$ dependence of the energy on the relative phase) may be regarded as equivalent to a derivation of the Josephson effect. At low temperatures, the collective dynamics of an inhomogeneous BEC is well described by the Gross-Pitaevskii (GP) Hamiltonian [10]

$$H[\Psi, \Psi^*] = \int d\mathbf{r} \left(\frac{\hbar^2}{2m} |\nabla \Psi(\mathbf{r})|^2 + V_{\rm ext}(\mathbf{r}) |\Psi(\mathbf{r})|^2 + \frac{g}{2} |\Psi(\mathbf{r})|^4 \right), \tag{6}$$

where $g = 4\pi\hbar^2 a/m$ (a is the s-wave scattering length). Neglecting depletion [11], the normalization can be taken as $\int d\mathbf{r}\rho = N_T$, being N_T the total number of atoms. The extremum condition $\delta(H - \mu_0 N_T)/\delta\Psi^* = 0$ (where μ_0 is the equilibrium chemical potential) leads to the well known GP equation [10].

II. PHASE RIGIDITY AND BROKEN GAUGE SYMMETRY

The assumption of a uniform phase within each Bose condensate is implicit in most discussions of the Josephson effect. It seems therefore convenient to check explicitly that this is a physically meaningful approximation. To prove it, let us estimate the energy of a one-radian fluctuation of the phase across the condensate near equilibrium in a single spherical harmonic well [see eq. (6)],

$$\frac{\hbar^2}{2m} \int d\mathbf{r} \rho |\nabla \phi|^2 \simeq \frac{\hbar^2}{2m} \frac{N_T}{4R^2} \simeq 0.7 N_T^{3/5} \frac{\hbar\omega_0}{2}, \tag{7}$$

where $R = a_0(15aN_T/a_0)^{1/5}$ is the cloud radius estimated within the TF approximation [9], ω_0 is the harmonic oscillator frequency of the well, and $a_0 = (\hbar/m\omega_0)^{1/2}$ is the oscillator length. For the last approximate equality we have used typical parameters $a = 5$ nm and $a_0 = 10^{-4}$ cm. The characteristic temperature of such an oscillation can be as big as 10 μK [7]. Since temperatures as low as 100 nK can be reached in current BEC setups, it is acceptable to view the phase of a condensate within one well as internally frozen and rigid. It is clear from (7) that spatial fluctuations of the phase are energetically more costly the higher the value of the condensate density ρ. For this same reason, in the interestitial region between two wells, where $\rho(\mathbf{r})$ decreases appreciably, spatial phase variations require less energy and are thus easier to activate at low temperatures. Here lies the essence of the low energy Josephson dynamics. At low temperatures, the picture emerges of two wells, each with an internally uniform phase, whose relative phase

$$\varphi = \int_1^2 d\mathbf{l} \cdot \nabla \phi, \tag{8}$$

induced by local variations of $\phi(\mathbf{r})$ in the interstitial region, is the only active dynamical variable.

From the theory of phase transitions, we know that some type of "rigidity" is generally exhibited in the ordered phase below the critical temperature [12]. In the case of superconductors, superfluids, and Bose-Einstein condensates (BEC's), one may speak of "phase rigidity" because, given an electromagnetic gauge (or its equivalent for superfluids and BEC's), the phase is determined everywhere once it has been fixed at a given point. Alternatively, since gauge and phase are intimately connected in quantum mechanics, we can say that, given a choice of phase, we have lost the freedom to choose the gauge, hence the term "gauge symmetry breaking".

Long range phase coherence can be destroyed by thermal or quantum fluctuations. A simple picture of how this occurs can be developed by analyzing the Josephson Hamiltonian (3), which may be viewed as a two-site reduction of the more general energy functional (6). In equilibrium, good phase coherence between sites 1 and 2 is obtained in the harmonic limit (4). Then it is appropriate to speak of global phase coherence or rigidity in the combined double BEC system. Thermal fluctuations randomize the phase for $k_B T \gtrsim E_J$. To see how quantum fluctuations can destroy the two-site phase coherence, we note that, since N and φ are canonically conjugate variables, N can be represented as $N = i\partial/\partial\varphi$ in a quantum description. As a consequence, the capacitive term in (3) plays the role of the kinetic energy for the phase variable. At zero temperature, quantum fluctuations can destroy phase rigidity between sites 1 and 2 if the interactions are sufficiently strong ($E_C \gg E_J$). In this limit, the relative particle number becomes a good quantum number [see eq. (3)] and the two condensates behave as essentially independent. Because of the weak connection, it is much easier to destroy the relative phase coherence between condensates 1 and 2, than the internal phase rigidity within each condensate. We have already proved that internal coherence cannot be destroyed by thermal fluctuations at the experimental temperatures. On the other hand, long range coherence within each condensate is preserved from destruction by quantum fluctuations thanks to the three-dimensional character of the problem, as stated by the Mermin-Wagner theorem in quantum statistical mechanics.

III. EXTERNAL JOSEPHSON EFFECT

A. Semiclassical description

Let us assume that $V_{\rm ext}(\mathbf{r})$ is such that our system has the structure of two weakly connected condensates 1 and 2. We assume that the condensates are confined within spherical harmonic wells of the same frequency ω_0. First we wish to analyse the semiclassical dynamics. Then, N and φ can be treated as simultaneously well-defined, and we may write for the wave function the ansatz [13]

$$\tilde{\Psi}(\mathbf{r},t) \sim \Psi_1(\mathbf{r}; N_T/2 - N(t)) + e^{i\varphi(t)}\Psi_2(\mathbf{r}; N_T/2 + N(t)), \tag{9}$$

where $\Psi_i(\mathbf{r};n)$ is the (real) equilibrium wave function for the isolated well i containing n bosons. It is straightforward to show that, to lowest order in the overlap integrals $\int \Psi_1\Psi_2$, the energy functional for $\tilde{\Psi}$ takes the form

$$H(\varphi, N) \simeq E_B(N) + E_J(N)\,(1-\cos\varphi)\,, \tag{10}$$

where $E_B(N)$ is the bulk energy of the two isolated wells with N transferred atoms, and $E_J(N)$ is the Josephson coupling energy.

In the TF limit [9], the bulk energy $E_B(N)$ is mostly due to interactions, and it may be expanded as

$$E_B(N) \simeq E_B(0) + \mu_0'(\frac{N_T}{2})N^2 + \frac{1}{12}\mu_0'''(\frac{N_T}{2})N^4. \tag{11}$$

Since $\mu_0 \sim N_T^{2/5}$ [9], the ratio between the third and second terms in the expansion is $0.32\,N^2/N_T^2$, which means that the last term can be neglected in a wide range of situations. To avoid complications stemming from possible resonances between Josephson oscillations (see below) and intrawell excitations, we require $\mu_0(N_T/2+N)-\mu_0(N_T/2-N) \ll \hbar\omega_0$, where we use the result that the first normal mode of a spherical well lies approximately at $\hbar\omega_0$ above the ground state [14,15]. This condition is realized when $N/N_T \ll 4.6 N_T^{-2/5}$ for typical parameters. This upper bound is of order $2-10\%$ for $N_T \sim 10^4 - 10^6$.

B. Josephson coupling energy in one dimension

The expression for the coupling energy $E_J(N)$ in terms of Ψ_1 and Ψ_2 implicit in (10) is rather complicated and difficult to handle, so it is desirable to find a simpler expression. Let us assume that the $(1-\cos\varphi)$ dependence for the energy of a Josephson link has already been proved by *e.g.* the arguments in the previous subsection, and focus on the particular case $\varphi \ll 1$, when $\Psi(\mathbf{r})$ is very close to the ground state wave function $\Psi_0(\mathbf{r})$ with $\varphi = 0$. In such a case, the integrand $|\nabla\Psi|^2$ in (6) is mostly due to phase variation. In one dimension, we can then write

$$E[\phi] \equiv \frac{\hbar^2}{2m}\int_1^2 dx\rho(x)\left|\phi'(x)\right|^2 \simeq \frac{1}{2}E_J\varphi^2, \tag{12}$$

where the phase term in (3) has been approximated quadratically, and where the integration extends between two points beyond which the phase is practically uniform.

From (1), we know that a supercurrent must flow between the two condensates whose value is

$$I \simeq (E_J/\hbar)\varphi. \tag{13}$$

On the other hand, particle number conservation requires

$$I = (\hbar/m)\rho(x)\phi'(x) = \text{cnst}. \tag{14}$$

Inserting the resulting expression for $\phi'(x)$ into (12), and using (13), one obtains a functional expression for the Josephson coupling energy in terms of the ground state superfluid density:

$$E_J = \frac{\hbar^2}{m}\left[\int_1^2 \frac{dx}{\rho_0(x)}\right]^{-1}, \tag{15}$$

where we have used the fact that, for $\varphi \ll 1$, $\rho \simeq \rho_0 \equiv |\Psi_0|^2$.

C. Three dimensions: Electrostatic analogy

We assume again that the $(1-\cos\varphi)$ dependence has already been proved and consider the case $\varphi \ll 1$. Analogously to (12), we can write

$$E[\phi] \equiv \frac{\hbar^2}{2m}\int_{\rm ext} d\mathbf{r}\rho(\mathbf{r})|\nabla\phi(\mathbf{r})|^2 \simeq \frac{1}{2}E_J\varphi^2, \tag{16}$$

where the integration extends over the region exterior to the condensates (the results are quite independent on the precise location of the condensate borders). The phase within the condensates is assumed to be uniform. The only way for eq. (16) to have such a dependence on the total phase difference φ and not on the details of $\nabla\varphi$ is that the condition $\delta E[\phi]/\delta\phi(\mathbf{r}) = (\hbar^2/2m)\nabla\cdot(\rho\nabla\phi) = 0$ is satisfied, something which is guaranteed by current conservation $[(\hbar/m)\rho\nabla\phi$ is the particle current density]. The alert reader may have noted that, except for trivial factors, this is the electrostatic equation for the electric displacement vector $\mathbf{D} \equiv \rho\nabla\varphi$ in a medium with a nonuniform dielectric constant $\rho(\mathbf{r})$. Boundary conditions for $\phi(\mathbf{r})$ are given by its value at the borders of each condensate (φ_1 and φ_2), which act as conductors in this analogy. We have a system of two conductors held at a potential difference φ and a dielectric medium surrounding them. Then $E[\phi]$ is essentially the energy of this capacitor. Potential theory tells us that $E[\phi] \sim \varphi^2$, with E_J in (16) playing the role of the mutual capacitance of the two conductors in the presence of such a dielectric.

Standard variational arguments can be invoked to prove that, for parallel plate boundary conditions [13],

$$\int \frac{dxdy}{\int_1^2 dz\rho(x,y,z)^{-1}} \leq \frac{mE_J}{\hbar^2} \leq \left[\int_1^2 \frac{dz}{\int dxdy\rho(x,y,z)}\right]^{-1} \tag{17}$$

(here z is the longitudinal coordinate connecting the centers of the two wells, and x,y are the transverse coordinates). The lower bound is obtained by removing the positive term $(\partial\phi/\partial x)^2+(\partial\varphi/\partial y)^2$ from the energy functional (16), while the upper bound is derived by taking φ independent of x,y. If the equilibrium density can be factorized in the region controlling the capacitance (a not very restrictive condition in practice; see above), we can write $\rho_0(\mathbf{r}) = f(x,y)g(z)$ and the two bounds become identical. Taking $\mathbf{r}=0$ at the middle point of the double well configuration, and choosing $x=y=0$ to be the line connecting the two well centers, we can finally write [13]

$$E_J = A\frac{\hbar^2}{m}\left[\int_1^2 \frac{dz}{\rho_0(0,0,z)}\right]^{-1}, \tag{18}$$

being $A = f(0,0)^{-1}\int dxdyf(x,y)$ an effective area. The factorization condition is much more general that it might appear at first sight, since it only has to be satisfied in the interstitial region near the top of the barrier giving the dominant contribution to E_J. There $V_{\rm ext}(\mathbf{r})$ has a saddle-point and thus it can be decoupled in longitudinal and transverse coordinates. Since interactions are negligible in the bottle neck region connecting the two wells, we conclude that the wave function must be factorizable.

D. WKB calculation

In order to proceed further, we need an estimate of $\rho_0 = |\Psi_0|^2$ in the region of interest. Here we reproduce sketchily the WKB calculation of ref. [13]. For two identical wells in equilibrium, the ground state wave function is symmetric in z. Focussing on the $x=y=0$ line giving the dominant contribution, we write

$$\Psi_0(z) = \frac{B}{\sqrt{p(z)}}\cosh\left[\frac{1}{\hbar}\int_0^z dz'p(z')\right] \tag{19}$$

where $p(z) = [2m(V_{\rm ext}(0,0,z)-\mu_0)]^{1/2}$ and B is a constant to be determined from the work of ref. [16]. Introducing (19) into (18) we obtain

$$E_J \simeq \frac{\hbar A|B|^2}{m}\left[2\tanh\left(\frac{S}{2}\right)\right]^{-1}, \tag{20}$$

where $S \equiv \int_1^2 p(z)dz/\hbar$ is the dimensionless instanton action. One finally gets [13]

$$E_J \simeq \frac{e^{-S}}{\tanh(S/2)} \left(\frac{N_T}{2}\right)^{1/3} \left(\frac{15a}{a_0}\right)^{-2/3} \frac{\hbar\omega_0}{2}. \tag{21}$$

The $N_T^{1/3}$ dependence can be understood qualitatively by noting [13,16] that $A \sim R^{2/3}$ and $B \sim R^{1/2}$. Since, in the TF approximation [9], $R \sim N_T^{1/5}$, one obtains $E_J \sim N_T^{1/3}$. Knowing that the critical temperature satisfies $k_B T_c \simeq N_T^{1/3}\hbar\omega_0$ [17], we may rewrite (21) as (for large S and typical parameters)

$$E_J \sim k_B T_c e^{-S} \tag{22}$$

which is reminiscent of the Ambegaokar-Baratoff formula for superconductors [2] if one notes that e^{-S} is the transmission amplitude for a particle to traverse the barrier.

E. Josephson plasma frequency

From (3) and (11), we have $E_C = 2\mu_0'(N_T/2)$, which, in the TF approximation [9] reads

$$E_C \simeq \frac{4}{5} \left(\frac{N_T}{2}\right)^{-3/5} \left(\frac{15a}{a_0}\right)^{2/5} \frac{\hbar\omega_0}{2}. \tag{23}$$

From (5), (21) and (23), we obtain

$$\omega_{JP} \simeq \left(\frac{2a_0}{15aN_T}\right)^{2/15} \frac{e^{-S/2}}{\sqrt{\tanh(S/2)}}\, \omega_0, \tag{24}$$

Since, usually, $\omega_0/2\pi \simeq 10 - 100$ Hz, we conclude that $\omega_{JP}/2\pi \lesssim 10$ Hz.

The ratio E_J/E_C is a good measure of the classical character of the relative phase φ. From (21) and (23), we find

$$\frac{E_J}{E_C} \simeq \left(\frac{2a_0}{15aN_T}\right)^{1/15} \frac{N_T a_0}{a} \frac{e^{-S}}{\tanh(S/2)}. \tag{25}$$

By varying S, the system can be driven from the classical regime ($E_J \gg E_C$) to the strong quantum limit ($E_J \ll E_C$). However, in the latter case, quantum effects are only important if we operate at ultralow temperatures $k_B T \lesssim E_C$. From (25), one concludes that, for typical numbers, S must be roughly $\gtrsim 10$ for quantum fluctuations to be important.

IV. EFFECT OF DISSIPATION

So far, we have described the dynamics of a conservative system. In real life, however, one should expect a certain amount of damping. The most obvious source of such damping is the incoherent exchange of normal atoms, and a quantitative discussion requires a generalization of our results to nonzero temperature. It is possible to perform a qualitative exploration of the effect of damping based on a simple transport model [13]. We can expect thermally excited normal atoms to follow the usual laws of dissipative currents. Specifically, they must give an Ohmic contribution to the current,

$$I_n = -G\mu, \tag{26}$$

where μ is a spontaneous or induced chemical potential difference (for simplicity, we assume equilibrium between condensate and thermal cloud within each well). As a result, the first Josephson equation is modified to read

$$\frac{d}{dt}N = \frac{E_J}{\hbar}\sin\varphi + I_n, \tag{27}$$

or, for small fluctuations [using (2) and (26)],

$$\ddot{\varphi} + GE_C\dot{\varphi} + \omega_{JP}^2\varphi = 0. \tag{28}$$

The classical harmonic oscillator described by eq. (28) is underdamped if

$$\sigma \equiv \frac{1}{\hbar G}\sqrt{\frac{E_J}{E_C}} \gg 1. \tag{29}$$

It only rests to calculate G for a specific system. A simple estimate can be made for the case in which the chemical potential lies near the top of the barrier, at a distance $V_1 \sim \hbar\omega_0/2$, so that $S \sim 1-5$. In this regime, we can still expect the WKB formula for E_J derived above to yield a reasonable approximation. However, if $k_BT \gg \hbar\omega_0/2 \sim V_1$, the thermal cloud lies mostly above the barrier and a radically different approach is needed to study its transport properties. For simplicity, we introduce the drastic approximation that particles impinging on the barrier with energy E are transmitted with probability one if $E > V_0$, and zero if $E < V_0$. Then, the flow of normal atoms due to a fluctuation in μ is only limited by the "contact resistance", a concept taken from ballistic transport in nanostructures [18]. We may write $G \equiv N_{\rm ch}/h$, where $N_{\rm ch}$ is an effective number of available transmissive channels. Within a continuum approximation, and assuming that only transverse channels with a minimum energy between μ_0 and $\mu_0 + k_BT$ are populated, we find (taking $\mu \ll k_BT$) $N_{\rm ch} = mA_nk_BT/2\pi\hbar^2$, where A_n is a mean transverse contact area seen by normal bosonic particles [13]. Approximating [11] $A_n \sim 2\pi k_BT/m\omega_0^2$, we find $N_{\rm ch} \sim (k_BT/\hbar\omega_0)^2$, and hence

$$\sigma \sim 2\pi N_0^{7/15}(\hbar\omega_0/k_BT)^2 e^{-S/2}. \tag{30}$$

Interestingly, the same result is formally obtained in the high barrier limit, where normal atom tunneling also lies in the WKB regime [13]. This gives additional confidence in the adequacy of the estimate (30). Taking $k_BT = 10\hbar\omega_0$, we find $\sigma \sim 0.38$ for $N \sim 10^4$, and 3.3 for $N \sim 10^6$, if $S \sim 5$. The equivalent numbers for $S \sim 1$ are 2.8 and 24. The conclusion is that coherent Josephson dynamics can be observed in current atomic Bose condensates if the barrier is low. Underdamped dynamics can be further favored by decreasing T and increasing N.

V. INTERNAL JOSEPHSON EFFECT

It is possible to couple different hyperfine states of an atom (say $|Fm>$ and $|F'm'>$) through laser light. In particular, by applying a laser pulse, one may force an atom initially in state $|Fm>$ to evolve in a controllable way towards another state $|F'm'>$. If such a pulse is applied to a macroscopic condensate, then it is the whole ensemble of atoms what evolves coherently. Thus it is possible to prepare a condensate in which each atom is in the same coherent superposition of the two states. Such an experiment has been realized by Hall *et al.* [19] with the $|1,-1>$ and $|2,1>$ states of ^{87}Rb (hereafter, $|A>$ and $|B>$). A natural variant of this experiment consists in using laser light which also induces the opposite rotation (*e.g.* linearly polarized light), in such a way that the two states are coherently connected in both directions, very much like Cooper pairs in a Josephson junction can tunnel from left to right and vice versa. In such a setup, it will be possible to study the novel and, potentially, extremely rich "internal" Josephson effect, which bears some analogies with the physics of superfluid ^{3}He-A, as noted by Leggett [20].

The analysis of the spatial (external) Josephson effect given in section III is based on the knowledge of $V_{\rm ext}(\mathbf{r})$, where $\mathbf{r}$ we can be varied continuosly between wells. A similar study of the internal Josephson effect would not be practical. Fortunately, all is really needed is the value of the matrix element connecting states $|A>$ and $|B>$, which can be known by other means. Then, it is most convenient to employ a two-site description of the Josephson link. We write [21,22]

$$H = -\frac{\hbar\omega_R}{2}(a^\dagger b + b^\dagger a) + \frac{u}{2}[(a^\dagger a)^2 + (b^\dagger b)^2], \tag{31}$$

where $a^\dagger(b^\dagger)$ creates one atom in state $|A>(|B>)$, u accounts for the interactions, and ω_R is the Rabi frequency. For simplicity, we assume that the two atomic energies, as well as the intraspecies interactions, are equal, and neglect the interspecies interaction. Particle number conservation requires $a^\dagger a + b^\dagger b = N_T \equiv 2N_0$.

To establish the connection with the pendulum Hamiltonian, we note that, since particle eigenstates admit a representation $\Phi_N(\varphi) = (2\pi)^{-1/2}\exp(-iN\varphi)$, we may write for (3)

$$< N+1|H|N >= -E_J/2. \tag{32}$$

A similar analysis for the two-site Hamiltonian (31) yields

$$\begin{aligned} < N+1|H|N > &= -(\hbar\omega_R/2) < N_A+1, N_B-1|a^\dagger b|N_A, N_B > \\ &= -(\hbar\omega_R/2)\sqrt{(N_A+1)N_B}. \end{aligned} \tag{33}$$

Noting that $N_{A,B} = N_0 \pm N$, and identifying (32) and (33), we arrive at

$$E_J(N) = \hbar\omega_R \left[N_0(N_0+1) - N(N+1)\right]^{1/2}. \tag{34}$$

In the limit $1 \ll N \ll N_0$, eq. (34) becomes

$$E_J = N_0 \hbar\omega_R, \tag{35}$$

which is a clear manifestation of the phenomenon of Bosonic amplification. The identification of (31) with the pendulum Hamiltonian is completed by noting that (apart from constant energy shifts) the interaction term can be written uN^2, *i.e.* $u = E_C/2$.

VI. CROSSOVER FROM JOSEPHSON TO RABI DYNAMICS

A clear limitation of the standard pendulum Hamiltonian (3) is that, in the non-interacting limit $E_C \to 0$, the dynamics is suppressed; in particular, the Josephson plasma frequency (5) for small collective oscillations vanishes. This contradicts our physical notion that noninteracting atoms should indeed exhibit some dynamics. For instance, under the effect of Hamiltonian (31), an atom initially prepared in state $|A>$ will undergo Rabi oscillations between states $|A>$ and $|B>$ with frequency ω_R. This effect is clearly beyond the scope of a rigid (with N-independent E_J) pendulum model such as (3). However, it should be describable by (31), which has a well-defined non-interacting limit. Thus, it seems of interest to develop a unified description which views collective Josephson behavior and single atom Rabi oscillations as particular cases of a more general dynamics. To that end, let us rewrite (34) in a semiclassical form,

$$E_J(N) = \hbar\omega_R \sqrt{N_0^2 - N^2}. \tag{36}$$

The classical dynamics of the resulting non-rigid pendulum Hamiltonian,

$$H = -N_0\hbar\omega_R \sqrt{1 - N^2/N_0^2}\cos\varphi + (E_C/2)N^2, \tag{37}$$

has been analyzed by Smerzi *et al.* [23]. In the limit of small φ and N, (37) acquires the harmonic form

$$H = N_0\hbar\omega_R \frac{\varphi^2}{2} + \left(E_C + \frac{\hbar\omega_R}{N_0}\right)\frac{N^2}{2}, \tag{38}$$

with a natural oscillation frequency

$$\begin{aligned} \omega^2 &= \omega_{JP}^2 + \omega_R^2 \\ &= (N_0 E_C/\hbar)\omega_R + \omega_R^2. \end{aligned} \tag{39}$$

It is thus clear that, for a given interaction strength, the double BEC system can be driven continuously from the Josephson to the Rabi regime by varying ω_R, something feasible with current laser technology. Noting that $E_C \sim \mu_0/N_0$, the Josephson limit corresponds to $\mu_0 \gg \omega_R$.

The difference between Josephson and Rabi dynamics is more dramatic when one considers large values of N, comparable to N_0. When $E_C = 0$, it is easy to see that (37) always yields oscillations of $N(t)$ around zero with frequency ω_R. By contrast, when $E_C \neq 0$, the frequency depends on the initial value $N(0)$ and, when the ratio $\Lambda \equiv N_0 E_C/\hbar\omega_R$ exceeds a critical value Λ_c, the oscillations average to a nonzero number [21,23]. This number Λ_c depends on $N(0)$ and $\varphi(0)$. In particular, $\Lambda_c = 2$ when $N(0)$ takes its maximum value N_0, and $\Lambda_c \to \infty$ when $N(0) \to 0$, *i.e.* in the small amplitude limit, there are always oscillations of zero average and frequency given by (39). The phenomenon by which, due to macroscopic quantum coherence, it is possible (for $\Lambda > \Lambda_c$) to create a large, long lived particle number imbalance between two connected BEC's, is called quantum self-trapping [21,23], and it is equivalent to the nonzero average angular momentum of a mechanical pendulum undergoing complete turns. Like the pendulum motion, the average particle imbalance can decay because of dissipation. Quantum self-trapping cannot be realized in superconducting Josephson junctions because, the chemical potential differences would have to be greater than the superconducting gap [23], thus allowing quasiparticles to intervene and complicate the dynamics. This is not an important limitation in the case of BEC's. There exists a close and well-studied analog of quantum self-trapping in the longitudinal NMR of ^{3}He-A [20].

Eq. (37) describes in a unified way Josephson and Rabi ($E_C = 0$) dynamics. This is analogous to how eq. (6) can yield both the Gross-Pitaevskii and the Schrödinger ($g = 0$) equations for the wave function of a many boson system. The crossover between collective Josephson and individual Rabi dynamics cannot be studied in superconductors and superfluids, because there interactions are never completely negligible. It is a nice feature of Bose-Einstein condensation that it will allow us to study the crossover between these two qualitatively different dynamical regimes in an elegant fashion.

VII. PHASE DYNAMICS OF SEPARATE CONDENSATES

The class of Josephson links describable by the harmonic approximation (4) includes many of the foreseeable connections that will be realized between BEC's. At equilibrium, the phase for such systems is narrowly peaked around $\varphi = 0$. Being canonically conjugate variables, N and φ satisfy the uncertainty relations $\Delta N \Delta\varphi \geq 1/2$. In general, their m.r.s. values are

$$\Delta\varphi = (E_i/E_J)^{1/2} \quad \Delta N = (E_i/E_C)^{1/2} \,, \tag{40}$$

being $E_i = (\hbar\omega_{JP}/2)\coth(\hbar\omega_{JP}/2k_BT)$ the average energy of the oscillator. If E_C and $k_BT \ll E_J$, we have $\Delta\varphi \ll 1$. It is then of interest to analyze the phase dynamics of two BEC's which, having formed an equilibrium Josephson link, are suddenly disconnected (E_J is made zero). Such a study may allow us to understand better the robustness of the phase definition when the support of the Josephson coupling is no longer present. One may naively think that two separate BEC's have a well-defined relative particle number and that the phase is necessarily random. However, if the starting point is that of a well-defined phase, the situation is much less clear, since phase randomization needs a finite time that may be sensitive to the details of the environment. After all, we are all familiar with physical variables (such as the position of macroscopic objects) which are permanently well-defined and yet are not constants of motion. There are at least three physical factors which favor this long term definition of non-conserved quantities: (i) large inertia (small effective E_C), (ii) high potential barriers in the effective potential landscape, and (iii) damping due to the environment. For the phase of a Josephson junction, at least the first two ingredients (and some times the three of them; see section IV) are present. Here we wish to understand what is the fate of the phase when the second factor (the Josephson coupling) is removed.

The randomization of the phase after suppression of the Josephson coupling was investigated in refs. [24] and [25] for the case of superconductors and superfluids. The possibility of performing interference experiments between BEC's [26,19] has revived the interest in the study of this [27] and related problems [28].

A. Ballistic randomization

In the absence of external perturbations, the evolution of the wave function at $t > 0$ is that of an initially narrow wave packet on a circle. Assuming for the moment that φ is an extended variable, the phase uncertainty at long times is

$$\Delta\varphi(t) \simeq \sqrt{E_i E_C}\, t/\hbar = \Delta N E_C t/\hbar \tag{41}$$

The ballistic time τ_B is defined as the time it takes $\Delta\varphi$ to be of order 2π. We get

$$\tau_B = h/\sqrt{E_i E_C}. \tag{42}$$

A typical number for superconductors can be $\tau_B \sim 10^{-8}$ s, while, for superfluids, the equivalent time may be of the order of minutes or hours, depending on whether the capacitive forces are due to compressibility (fixed volume) or gravity [24,25]. The randomization of the relative phase between BEC's lies in an intermediate time scale. Using (23), we find $\tau_B \sim 20 - 100$ ms (with $N \sim 10^4 - 10^6$) for $T = 100$ nK.

B. Recurrent dynamics

The phase is actually an angular variable for which 0 and 2π are to be identified and this implies that N has integer eigenvalues. When $E_J = 0$, the wave function evolves as

$$\Phi(\varphi,t) =< \varphi|\Phi(t) >= \sum_N a_N \Phi_N(\varphi) \exp(-iE_C N^2 t/2\hbar), \tag{43}$$

where $a_N =< N|\Phi(0) >$. From (43), we notice the existence of a recurrence time [25]

$$\tau_R = 2h/E_C, \tag{44}$$

after which the wave packet recombines, $\Phi(\varphi, \tau_R) = \Phi(\varphi, 0)$. The property $E_i \ll E_J$ implies $\tau_R \gg \tau_B$, so that the two time scales are well separated. This observation renders meaning to the apparently oversimplified mechanism of ballistic spreading, provided one considers times much smaller than τ_R.

A detailed analysis reveals the following picture [29]: An initially narrow wave packet begins to broaden according to (41). After a time of order τ_B, the wave packet is almost uniformly spread around the circle, apart from fast oscillations. At a much later time τ_R, the wave packet recombines and broadens again in a time scale τ_B. A typical value may be $\tau_R \sim 10^{-5}$ s for a superconductor, and an astronomically large one for a superfluid. Again, BEC's lie somewhere in between: We find $\tau_R \sim 30 - 500$ s for $N \sim 10^4 - 10^6$. It must be noted that recurrence would only be observed in quite ideal conditions (very low noise, good time resolution, etc.), perhaps not easily realizable.

C. Diffusive randomization

So far we have neglected the possible effect of dissipation due to the environment. In refs. [25,29], a source of noise was studied which, when really present, can be expected to dominate the dynamics, namely, the exchange of particles with a normal reservoir. In BEC's, such a reservoir could be provided by the thermal cloud.

Following ref. [25], we base our study on a useful analogy which can save us a good amount of work. If the contact between the condensate and the reservoir is Ohmic, *i.e.* the rate of particle exchange behaves as

$$\delta \dot{N} = -\Gamma \delta\mu. \tag{45}$$

where $\delta\mu$ is a possible chemical potential difference between condensate and thermal cloud, then our problem is formally identical to that of a Brownian particle (BP). The phase plays the role of the position and the capacitance is analogous to the mass. The particle number N is equivalent to the momentum p (both are conserved quantities that are exchanged with the bath), and the chemical potential difference $\mu \sim \dot{\varphi}$ is the the analog of the particle velocity v (the relative position of two BP's behaves like another BP). The friction coefficient η, defined as $\dot{p} = -\eta v$ (where v is a deviation of the velocity from its zero equilibrium value), is equivalent to the coefficient Γ.

Once the analogy with the BP has been established, we may borrow from the classical theory [30] and write for the evolution of the phase uncertainty

$$\Delta\varphi^2(t) = \frac{2k_BT}{\hbar^2\Gamma}\left[t - \frac{1}{\gamma}(1-e^{-\gamma t})\right], \tag{46}$$

where $\gamma = \Gamma E_C$. At short times ($\gamma t \ll 1$), we reproduce the ballistic randomization given by (41). By contrast, at long times ($\gamma t \gg 1$), the randomization becomes diffusive,

$$\Delta\varphi(t) \simeq \left(2k_BTt/\hbar^2\Gamma\right)^{1/2}. \tag{47}$$

Eq. (47) is a classical result. However, thanks to the work of ref. [31] on a quantum BP with Ohmic dissipation, we know that the diffusive law (47) applies at long times $t \gg \hbar/k_BT \gg \gamma^{-1}$, provided $\Delta\varphi(0) \ll 1$

Analogously to the ballistic case, it is possible to define a time for diffusive randomization

$$\tau_D = h^2\Gamma/2k_BT \tag{48}$$

For superconductors, τ_D may be of the order of minutes, a surprisingly long time if we compare it with the estimates for τ_B. This is a very important effect. The large disparity between the two time scales may be understood by noting that, while the fast ballistic randomization is due to the large value of E_C (small electrostatic capacitance), the diffusive dynamics (which takes over at a time $\gamma^{-1} \ll \tau_B$) is essentially independent of it and is determined instead by the coupling to the normal reservoir. We arrive at the apparently paradoxical result that, upon suppression of the Josephson contact, dissipation may help to preserve phase coherence.

D. Common vs. independent thermal clouds

It is important to emphasize that the diffusive mechanism described above only applies if the normal particle reservoir is *common* to both condensates. In VII.C, we have considered a situation in which, although the two condensates have been separated, the two thermal clouds remain connected to the point of behaving as a single one. In superconductors, an equivalent arrangement is not fundamentally difficult (the reservoir would be a normal metal). In the case of BEC's, we may have a common thermal cloud when the increase in the barrier height is large enough to render the two condensates unconnected, but still sufficiently small to allow for an essentially single thermal cloud.

If the two reservoirs are independent, then, following the BP analogy (in which the finite masses of each of the separate baths would cause a ballistic spread of the uncertainty in the relative position), the finite effective value of the total E_C for each one of the condensate plus reservoir system will give rise to a ballistic randomization of the relative phase. This seems to be the case in interference experiments between different hyperfine states [19].

The crucial difference between ballistic and diffusive spreading of the phase is that, in the former case, there is no restoring force for fluctuations in N. Therefore, any initial deviation ΔN gives rise to a permanent $\Delta\mu$ which causes a ballistic spread of the phase [see (41)]. On the contrary, if randomization is diffusive, an initial departure from equilibrium is quickly compensated and, in the average, $N(t)$ and $\mu(t)$ change sign many times, in such a way that $\Delta\varphi(t)$ grows slowly [see (47)].

VIII. CONCLUSIONS

I have presented an overview of the physics of the Josephson effect betwee Bose condensed systems, with emphasis on the recently achieved BEC's in trapped alkali gases. I have mostly focussed on those physical phenomena that are likely to be observed only (or more easily) in these novel systems. So I have omitted the discussion of problems (such as *e.g.* steady particle flow under the action of an alternating chemical potential) which may be viewed as straightforward applications of well known Josephson physics [2]. I have tried to underline the potential richness of the physics displayed by weakly connected BEC's. One may have an external (spatial) and an internal (hyperfine) Josephson effect. It seems possible to explore the crossover between collective Josephson behavior and independent boson Rabi dynamics. Finally, the observation of fascinating phenomena such as quantum self-trapping and macroscopic interference between separate Bose condensates seems also within reach of the emerging BEC technology. Everything indicates that the experimental and theoretical study of the Josephson effect between Bose-Einstein condensates will lead us to the exploration of most exciting new physics.

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[1] JOSEPHSON B.D., *Phys. Lett.*, **1** (1962) 251.
[2] BARONE A. and PATERNÒ B., *Physics and applications of the Josephson effect* (John Wiley & Sons, New York) 1982.
[3] AVENEL O. and VAROQUAUX E., *Phys. Rev. Lett.*, **55** (1985) 2704.
[4] ANDERSON M.H., ENSHER J.R., MATTHEWS M.R., WIEMAN C.E. and CORNELL E.A., *Science*, **269** (1995) 198.
[5] BRADLEY C.C, SACKETT C.A., TOLLETT J.J., and HULET R.G., *Phys. Rev. Lett.*, **75** (1995) 1786.
[6] DAVIS K.B., MEWES M.O., ANDREWS M.R., VAN DRUTEN N.J., DURFEE D.S., KURN D.M., and KETTERLE W., *Phys. Rev. Lett.*, **75** (1995) 3969.
[7] MEWES M.O., ANDREWS M.R., VAN DRUTEN N.J., KURN D.M., DURFEE D.S., TOWNSEND C.G., and KETTERLE W., *Phys. Rev. Lett.*, **77** (1996) 416; **77** (1996) 988.
[8] FEYNMAN R.P., LEIGHTON R.B., and SANDS M., *The Feynman Lectures on Physics* (Addison-Wesley Pub. Co., Reading) 1963.
[9] BAYM G. and PETHICK C.J., *Phys. Rev. Lett.*, **76** (1996) 6.

[10] PITAEVSKII L.P., *Zh. Eksp. Teor. Fiz.*, **40**, (1961) 646 [*Sov. Phys. JETP*, **13** (1961) 451]; GROSS E.P., *Nuovo Cimento*, **20** (1961) 454; *J. Math. Phys.*, **4** (1963) 195.
[11] GIORGINI S., PITAEVSKII L.P., and STRINGARI S., *Phys. Rev. Lett.*, **78** (1997) 3987.
[12] ANDERSON P.W., *Basic Notions of Condensed Matter Physics*, (Benjamin-Cummings, Menlo Park) 1984.
[13] ZAPATA I., SOLS F., and LEGGETT A.J., *Phys. Rev. A*, **57** (1998) R28.
[14] STRINGARI S., *Phys. Rev. Lett.*, **77** (1996) 2360.
[15] EDWARDS M., RUPRECHT P.A., BURNETT K., DODD R.J., and CLARK C.W., *Phys. Rev. Lett.*, **77** (1996) 1671.
[16] DALFOVO F., PITAEVSKII L.P. and STRINGARI S., *Phys. Rev. A*, **54** (1996) 4213.
[17] GIORGINI S., PITAEVSKII L.P., and STRINGARI S., *Phys. Rev. A*, **54** (1996) R4633.
[18] IMRY Y., in *Directions in Condensed Matter*, edited by G. GRINSTEIN and E. MAZENKO (World Scientific, Singapore) 1986.
[19] HALL D.S., MATTHEWS M.R., WIEMAN C.E., and CORNELL E.A., cond-mat/9805327.
[20] LEGGETT A.J., *Proc. ICAP XVI*, edited by W.E. BAYLIS, AIP Press, to be published.
[21] MILBURN G.J., CORNEY J., WRIGHT E.M., and WALLS D.F., *Phys. Rev. A*, **55** (1997) 4318.
[22] VILLAIN P., LEWENSTEIN M., DUM R., CASTIN Y., YOU L., IMAMOGLU A., and KENNEDY T.A.B., *J. Mod. Opt.*, **44** (1997) 1775.
[23] SMERZI A., FANTONI S., GIOVANAZZI S., and SHENOY S.R., *Phys. Rev. Lett.*, **79** (1997) 4950.
[24] LEGGETT A.J. and SOLS F., *Found. Phys.*, **21** (1991) 353.
[25] SOLS F., *Physica B*, **194-196** (1994) 1389.
[26] ANDREWS M.R., TOWNSEND C.G., MIESSNER H.J., DURFEE M.S., KURN D.M., and KETTERLE W., *Science*, **275** (1997) 637.
[27] BARNETT S.M., BURNETT K., and VACCARO J.A., *J. Res. Natl. Inst. Stand. Technol.*, **101** (1996) 593; IMAMOGLU A., LEWENSTEIN M., and YOU L., *Phys. Rev. Lett.*, **78** (1997) 2511; JAVANAINEN J. and WILKENS M., *Phys. Rev. Lett.*, **78** (1997) 4675, and *Phys. Rev. Lett.*, **81** (1998) 1345; LEGGETT A.J. and SOLS F., *Phys. Rev. Lett.*, **81** (1998) 1344.
[28] JAVANAINEN J. and YOO S.M., *Phys. Rev. Lett.*, **76** (1996) 161; WRIGHT E.M., WALLS D.F., and GARRISON J.C., *Phys. Rev. Lett.*, **77** (1996) 2158; LEWENSTEIN M. and YOU M., *Phys. Rev. Lett.*, **77** (1996) 3489; NARASCHEWSKI M., WALLIS H., SCHENZLE A., CIRAC J.I., and ZOLLER P., *Phys. Rev. A*, **54** (1996) 2185; CIRAC J.I., GARDINER C.W., NARASCHEWSKI M., and ZOLLER P., *Phys. Rev. A*, **54** (1996) R3714; WALLIS H., RÖHRL A., NARASCHEWSKI M., SCHENZLE A., and MIESSNER H.J., *J. Mod. Opt.*, **44** (1997) 1751; CASTIN Y. and DALIBARD J., *Phys. Rev. A*, **55** (1997) 4330;
[29] SOLS F. and HEGSTROM R.A., in *Fundamental Problems in Quantum Physics*, edited by M. FERRERO and A. VAN DER MERWE (Kluwer, Dordrecht) 1995.
[30] REIF F., *Foundations of Statistical and Thermal Physics* (McGraw-Hill, New York) 1965.
[31] HAKIM V. and AMBEGAOKAR V., *Phys. Rev. B*, **32** (1984) 423.